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An Analytic Perspective of Modified Wavelet Transform in Distributional Spaces

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ABSTRACT: In this paper, we investigate the analytic perspective of a generalized modified wavelet transform in the framework of generalized functions. The transform is defined with the help of a suitable kernel on the testing function space $E(\mathbb{R}^n)$ and is subsequently extended to its dual space $E^*(\mathbb{R}^n)$. We study the dependence of the transform on the complex variable p and establish its analyticity under an appropriate support condition on the generalized function. The proof is based on estimates of the derivatives of the kernel together with Cauchy's integral formula. These estimates show that the difference quotient converges to zero in the testing function space, allowing differentiation to be carried out within the distributional pairing. The same argument can then be applied repeatedly to obtain derivatives of arbitrary order. Thus, the generalized modified wavelet transform is shown to be analytic on $\mathbb{C}^n$, providing a useful analytical basis for further study of its properties in the distributional setting.

KEYWORDS: Fractional Fourier transform, wavelet transform, modified wavelet transform, generalized functions, testing function space, analyticity.

I. INTRODUCTION

Integral transforms are crucial for analyzing functions and signals. They have received a lot of attention in both pure and applied mathematics. The Classical Fourier Transform offers a global frequency representation but struggles with non-stationary or localized phenomena. This drawback has led to the creation of various generalizations, with fractional transforms gaining significant interest.

The FrFT builds on the classical Fourier transform by adding a fractional parameter. This allows for a continuous shift between the time and frequency domains, resulting in a more detailed analytical structure. Because of this flexibility, researchers have explored fractional transforms and their generalizations extensively.

In contrast, the wavelet transform is a useful tool for local signal analysis. It uses dilation and translation parameters, achieving joint time and frequency localization. This feature has sparked the development of many modified and generalized wavelet-type transforms. The interplay between fractional transforms and wavelet analysis naturally leads to new constructions that mix fractional order parameters with localization properties. Mathematically, it is essential to study these transforms within a solid functional analytic framework. This ensures they are properly defined and helps investigate their analytical properties.

The theory of generalized functions offers a fitting environment for this study. By defining transforms on suitable testing function spaces and extending them to their dual spaces, one can systematically look at properties like continuity, differentiability and analyticity. Within this framework, a transform's behavior closely relates to its associated kernel.

The originality of this work lies in the thorough examination of the analyticity of a generalized modified wavelet transform tied to a specific kernel in the context of testing function spaces and distributions. Unlike previous research that mainly addresses fractional or wavelet-type transforms within classical function spaces, this approach establishes analyticity directly in the dual space $E^*(\mathbb{R}^n)$. It does so through precise kernel estimates and Cauchy-type integral techniques. These findings provide a fresh analytical perspective on modified wavelet transforms and broaden their use in areas like signal processing and differential equations.

Preliminaries

1.1 The Modified Wavelet Transform

The Modified Wavelet Transform of f is defined by,

$$mWTf(x) = \int_{-\infty}^{\infty} Q C_{1\alpha} \exp[i C_{2\alpha}((x^2 + p^2)\cos\alpha - 2xp)] f(x) dx$$

$$\text{Where, } Q = \frac{\exp(\frac{i}{2}p^2 \tan^2 \alpha)}{C_{1\alpha}|a|^{1/2}}, \quad C_{1\alpha} = \frac{\exp(\frac{i}{2}b^2 \sin^2 \alpha)}{|\tan^{1/2} \alpha|^n}, \quad C_{2\alpha} = \frac{1}{2\sin\alpha}$$

$$a = \tan^{1/2} \alpha, \quad b = p \sec \alpha.$$

1.2 Distributional Generalized Modified Wavelet Transform

The distributional generalized modified wavelet transform of a function $f(x) \in E^*(\mathbb{R})$ is defined by,

$$mWT\{f(x)\}(p) = mW_{\alpha}(a, b) = \langle f(x), K_{\alpha}(x, p) \rangle$$

where the kernel $K_{\alpha}(x, p)$ is given by,

$$K_{\alpha}(x, p) = Q C_{1\alpha} \exp[i C_{2\alpha}\{(x^2 + p^2)\cos\alpha - 2xp\}],$$

$$\text{Where } Q = \frac{\exp(\frac{i}{2}p^2 \tan^2 \alpha)}{C_{1\alpha}|a|^{1/2}}, \quad C_{1\alpha} = \frac{\exp(\frac{i}{2}b^2 \sin^2 \alpha)}{|\tan^{1/2} \alpha|^n}, \quad C_{2\alpha} = \frac{1}{2\sin\alpha}$$

$$a = \tan^{1/2} \alpha, \quad b = p \sec \alpha$$

The right-hand side of the above equation is meaningful since $K_{\alpha}(x, p)$ belongs to the testing function space $E(\mathbb{R})$ as a function of x and $f(x) \in E^*(R)$. Therefore, the distributional pairing $\langle f(x), K_{\alpha}(x, p) \rangle$ is well-defined.

II. ANALYTICITY OF GENERALIZED MODIFIED WAVELET TRANSFORM

In this section, we show that the Generalized Modified Wavelet Transform is analytic in terms of its definition on the space of generalized functions.

2.1 Analyticity Theorem

Statement :

Let $f(x) \in E^*(\mathbb{R}^n)$ and let its generalized Modified Wavelet transform be defined by,

$$mWT\{f(x)\}(p) = mW_{\alpha}(a, b) = \langle f(x), K_{\alpha}(x, p) \rangle$$

Then $mW_{\alpha}(a, b)$ is analytic as a function of p belonging to $\mathbb{C}_n$, if $\text{supp } f \subset S_q$,

$$S_q = \{x: x \in \mathbb{R}^n, |x| \leq q, q > 0\}.$$

Moreover, $mW_{\alpha}(a, b)$ is differentiable and $D_p^{\ell}\{mW_{\alpha}(a, b)\} = \langle f(x), D_p^{\ell}K_{\alpha}(x, p) \rangle$.

Proof :

Let $p = (p_1, p_2, \dots, p_j, \dots, p_n) \in \mathbb{C}_n$ be arbitrary but fixed point. We first prove that

$$\frac{\partial}{\partial p_j}(mW_{\alpha}(p)) = \langle f(x), \frac{\partial}{\partial p_j}K_{\alpha}(x, p) \rangle.$$

For fixed $p_j \neq 0$, choose two concentric circles C and C' with centre at p_j and with radii r and r_1 respectively such that $0 < r < r_1 < |p_j|$.

Let Δp_j be a complex increment satisfying $0 < |\Delta p_j| < r$.

Consider,

$$\frac{mW_\alpha(p_j + \Delta p_j) - mW_\alpha(p_j)}{\Delta p_j} - \left\langle f(x), \frac{\partial}{\partial p_j} K_\alpha(x, p) \right\rangle = \left\langle f(x), \psi_{\Delta p_j}(x) \right\rangle \quad \dots(2.1.1)$$

We have,

$$\left\langle f(x), \psi_{\Delta p_j}(x) \right\rangle = \frac{1}{\Delta p_j} \{ \langle f(x), K_\alpha(x, p_j + \Delta p_j) \rangle - \langle f(x), K_\alpha(x, p_j) \rangle \} - \left\langle f(x), \frac{\partial}{\partial p_j} K_\alpha(x, p) \right\rangle$$

Using the linearity of the distributional pairing, this may be written as

$$\left\langle f(x), \psi_{\Delta p_j}(x) \right\rangle = \left\langle f(x), \frac{1}{\Delta p_j} \{ K_\alpha(x, p_j + \Delta p_j) - K_\alpha(x, p_j) \} - \frac{\partial}{\partial p_j} K_\alpha(x, p) \right\rangle,$$

Where $\psi_{\Delta p_j}(x) = \frac{1}{\Delta p_j} \{ K_\alpha(x, p_j + \Delta p_j) - K_\alpha(x, p_j) \} - \frac{\partial}{\partial p_j} K_\alpha(x, p)$.

For any fixed $x \in \mathbb{R}^n$ and fixed integer $l = (l_1, l_2, l_3, \dots, l_n)$, we compute

$$D_x^\ell \{ K_\alpha(x, p) \} = D_x^\ell \{ Q C_{1\alpha} e^{(iC_{2\alpha}[(x^2 + p^2) \cos \alpha - 2xp])} \}$$

Since Q and $C_{1\alpha}$ are independent of x , we obtain

$$D_x^\ell \{ K_\alpha(x, p) \} = Q C_{1\alpha} D_x^\ell \{ e^{(iC_{2\alpha}[(x^2 + p^2) \cos \alpha - 2xp])} \}$$

For any integer $\ell \geq 0$, the ℓ -th order partial derivative of the kernel with respect to x is given by

$$D_x^\ell \{ K_\alpha(x, p) \} = Q C_{1\alpha} e^{(iC_{2\alpha}[(x^2 + p^2) \cos \alpha - 2xp])} \\ \times \left[\sum_{k=0}^{\ell/2} \frac{\ell!}{k!(\ell-2k)!} \left(\frac{i}{\sin \alpha} \right)^{(\ell-k)} \left(\frac{\cos \alpha}{2} \right)^k (x \cos \alpha - p)^{(\ell-2k)} \right]$$

Since $K_\alpha(x, p) = Q C_{1\alpha} e^{(iC_{2\alpha}[(x^2 + p^2) \cos \alpha - 2xp])}$ and $2iC_{2\alpha} = \frac{i}{\sin \alpha}$,

the above expression may be written in the factored form

$$D_x^\ell \{ K_\alpha(x, p) \} = \left[\sum_{k=0}^{\lfloor \ell/2 \rfloor} \frac{\ell!}{k!(\ell-2k)!} (2iC_{2\alpha})^{\ell-k} \left(\frac{\cos \alpha}{2} \right)^k (x \cos \alpha - p)^{\ell-2k} \right] K_\alpha(x, p) \quad \dots(2.1.2)$$

Hence, for each fixed $x \in \mathbb{R}^n$, for fixed l and $0 < \alpha \leq \frac{\pi}{2}$,

$D_x^\ell \{ K_\alpha(x, p) \}$ is analytic inside and on C .

We have by Cauchy's integral formula,

$$D_x^\ell \psi_{\Delta p_j}(x) = \frac{1}{2\pi i} D_x^\ell \int_C K_\alpha(x, \tilde{p}) \left[\frac{1}{\Delta p_j} \left(\frac{1}{z - p_j - \Delta p_j} - \frac{1}{z - p_j} \right) - \left(\frac{1}{z - p_j} \right)^2 \right] dz,$$

Where $\tilde{p} = (p_1, p_2, \dots, p_{j-1}, z, p_{j+1}, \dots, p_n)$

$$\begin{aligned} &= \frac{1}{2\pi i} D_x^\ell \int_C K_\alpha(x, \tilde{p}) \left[\left(\frac{1}{(z - p_j - \Delta p_j)(z - p_j)} \right) - \left(\frac{1}{z - p_j} \right)^2 \right] dz \\ &= \frac{\Delta p_j}{2\pi i} D_x^\ell \int_C \left[\frac{K_\alpha(x, \tilde{p})}{(z - p_j - \Delta p_j)(z - p_j)^2} \right] dz \\ &= \frac{\Delta p_j}{2\pi i} \int_C \left[\frac{D_x^\ell K_\alpha(x, \tilde{p})}{(z - p_j - \Delta p_j)(z - p_j)^2} \right] dz \end{aligned}$$

Let $Q(x, \tilde{p}) = D_x^\ell K_\alpha(x, \tilde{p})$. Then

$$D_x^\ell \psi_{\Delta p_j}(x) = \frac{\Delta p_j}{2\pi i} \int_C \left[\frac{Q(x, \tilde{p})}{(z - p_j - \Delta p_j)(z - p_j)^2} \right] dz \quad \dots(2.1.3)$$

For all $z \in C$ and x restricted to a compact subset of $\mathbb{R}^n$, $0 < \alpha \leq \frac{\pi}{2}$,

$Q(x, \tilde{p}) = D_x^\ell k_\alpha(x, \tilde{p})$ as in (2.1.3) is bounded by a constant $L > 0$.

Moreover, $|z - p_j - \Delta p_j| \geq r_1 - r > 0$ and $|z - p_j| = r_1$. Therefore,

$$\begin{aligned} |D_x^\ell \psi_{\Delta p_j}(x)| &= \left| \frac{\Delta p_j}{2\pi i} \int_C \left[\frac{Q(x, \tilde{p})}{(z - p_j - \Delta p_j)(z - p_j)^2} \right] dz \right| \\ &\leq \frac{|\Delta p_j|}{2\pi} \int_C \frac{L}{(r_1 - r)r_1^2} |dz| \\ &\leq \frac{|\Delta p_j|}{2\pi} \frac{L}{(r_1 - r)r_1^2} 2\pi r_1 \\ |D_x^\ell \psi_{\Delta p_j}(x)| &\leq \frac{|\Delta p_j|L}{(r_1 - r)r_1} \quad \dots\dots(2.1.4) \end{aligned}$$

As $|\Delta p_j| \rightarrow 0$, the derivatives of $\psi_{\Delta p_j}$ tend uniformly to zero on compact subsets of $\mathbb{R}^n$.

Hence, $\psi_{\Delta p_j}(x)$ converges to zero in $E(\mathbb{R}^n)$. Since $f(x) \in E^*(\mathbb{R}^n)$, it follows that

$$\langle f(x), \psi_{\Delta p_j}(x) \rangle \rightarrow 0$$

Therefore, $mW_\alpha(a, b)$ is differentiable with respect to p_j . But this holds for $j = 1, 2, \dots, n$,

therefore the function $mW_\alpha(a, b)$ is analytic on $\mathbb{C}^n$.

Moreover, $D_p^\ell \{mW_\alpha(a, b)\} = \langle f(x), D_p^\ell K_\alpha(x, p) \rangle$.

This completes the proof.

III. CONCLUSION

In this work, the analyticity of the generalized modified wavelet transform associated with the prescribed kernel has been established in the distributional space $E^*(\mathbb{R}^n)$. Under the support condition $\text{supp } f \subset S_q$, we have shown that $mW_\alpha(a, b)$ is analytic with respect to $p \in \mathbb{C}^n$. The result follows by estimating the derivatives of the kernel and applying Cauchy's integral formula. These estimates show that the difference quotient tends to zero in the testing function space $E(\mathbb{R}^n)$, which allows differentiation to be performed within the distributional pairing. Therefore, the obtained result gives a useful analytical basis for studying further properties of the generalized modified wavelet transform in the distributional setting.

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